CS 237: Probability in Computing

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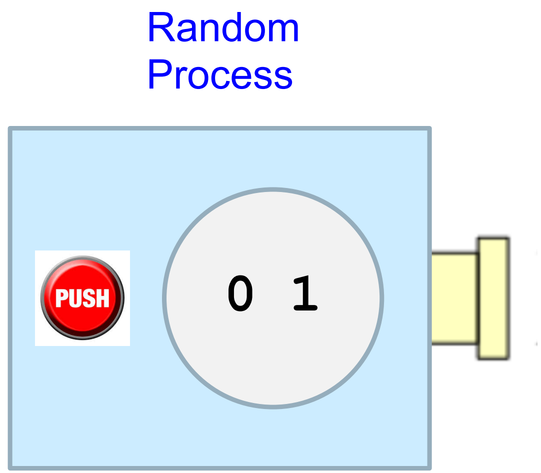
Lecture 1: Basic Definitions and Concepts of Probability Theory

- Random Experiments
- Sample Points, Sample Spaces, Events
- Probability Functions and Probability Spaces
- Classification of Probability Spaces:
  - Discrete (Finite or Countably Infinite)
  - Continuous (Uncountably Infinite)
- Axioms for Probability

Random Experiments

A Random Experiment is a process that produces uncertain outcomes from a well-defined set of possible outcomes. Usually there is some kind of physical experiment which is being modeled theoretically.



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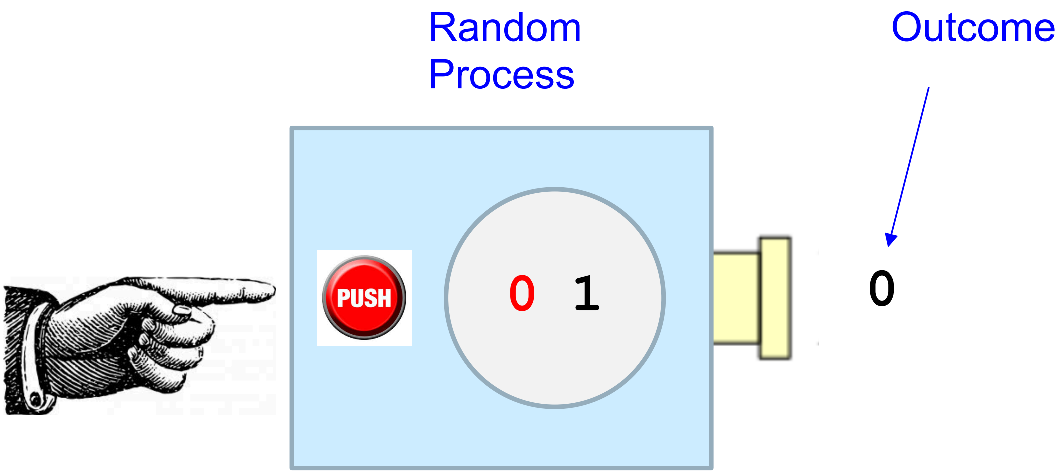
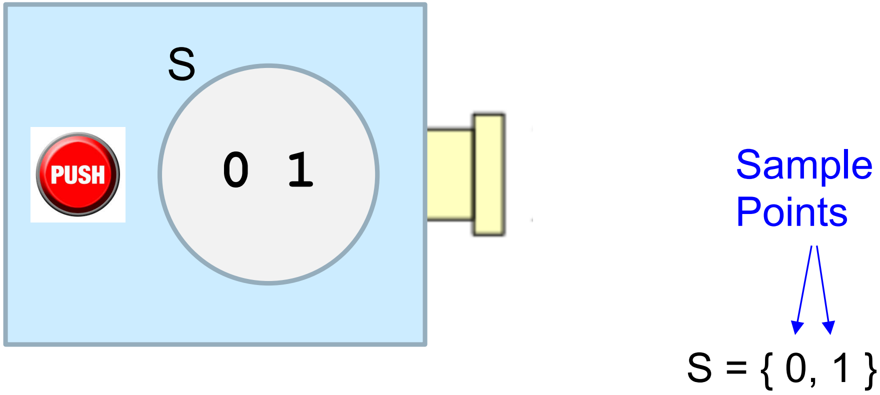


Random Experiments, Sample Spaces and Sample Points

The Sample Space (denoted by S or sometimes the Greek letters Σ or Ω) of the experiment **is the set of possible outcomes**, and any individual outcomes is called a Sample Point. Sample points can be any discrete objects.



Random Experiments, Sample Spaces and Sample Points

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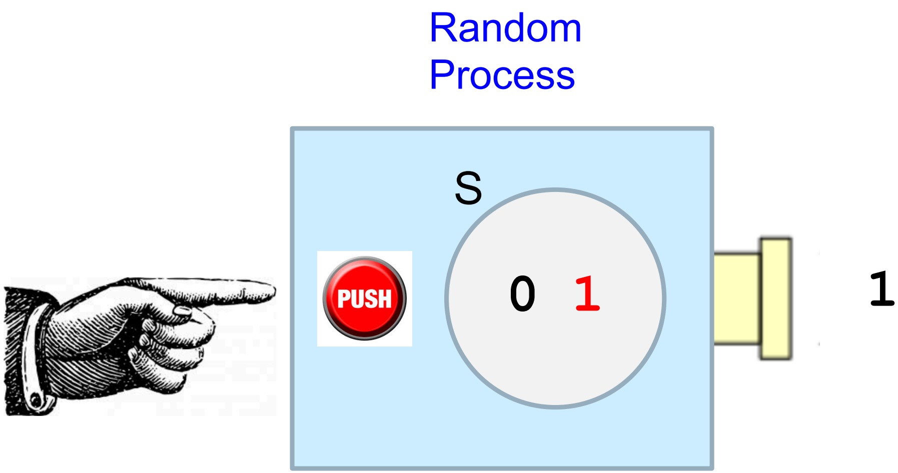


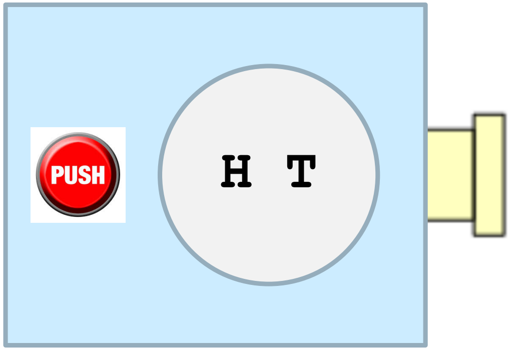
S = { 0, 1 }

Random Experiments, Sample Spaces and Sample Points

Often such an experiment represents some actual physical experiment.

Example: Flip a coin, is it heads or tails?

Random
Process



?

S = { H, T }

Random Experiments, Sample Spaces and Sample Points

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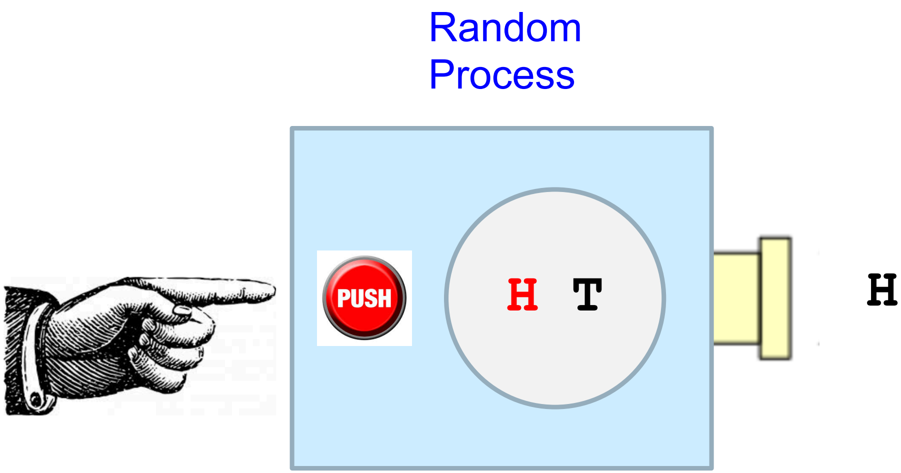
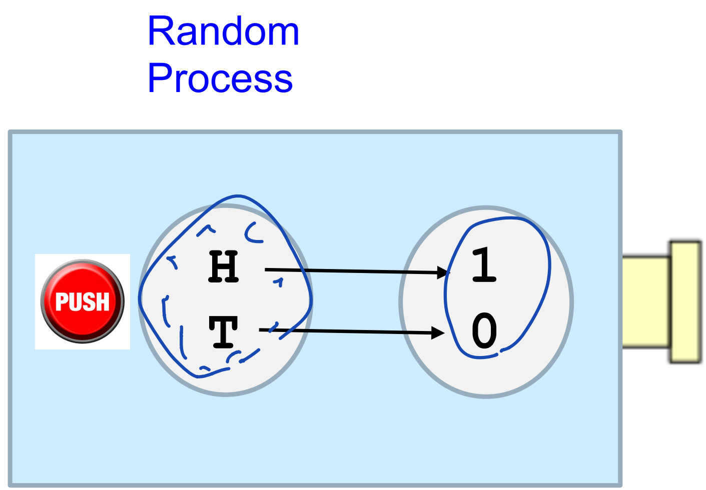


$S = \{ H, T \}$

Random Experiments, Sample Spaces and Sample Points

The description of the random experiment may indicate that there is more than one stage to creating the outcome, for example, a physical analogy may produce results that have to be interpreted as "outcomes."

Example: Flip a coin, count how many heads show.



S = { 1, 0 }

Random Experiments, Sample Spaces and Sample Points

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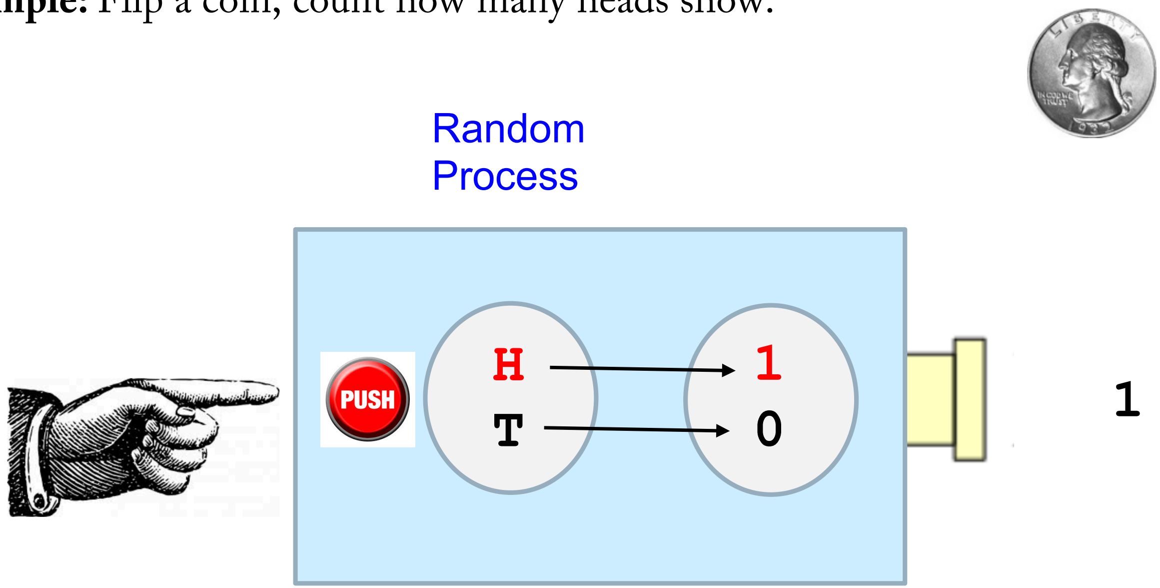

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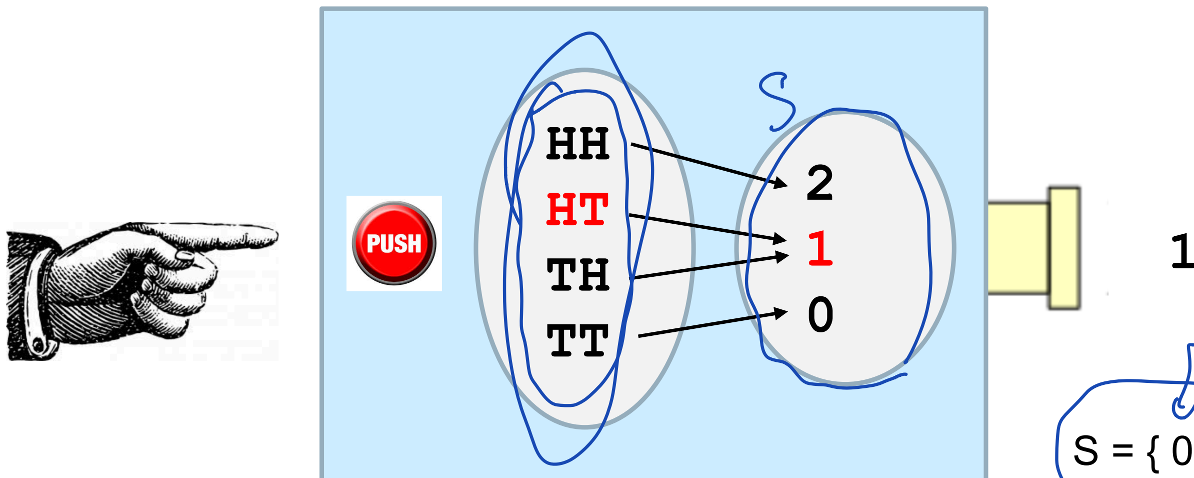
Random Experiments, Sample Spaces and Sample Points

The description of the random experiment may indicate that there is more than one stage to creating the outcome, for example, a physical analogy may produce results that have to be interpreted as "outcomes."

Example: Flip two coins, count how many heads show.



S = { 0, 1, 2 }

Sample Spaces

= ALL POSSIBLE OUTCOMES / SAMPLE POINTS

The **Sample Points** can be just about anything (numbers, letters, words, people, etc.) and the **Sample Space** (= **any** set of sample points) can be

Finite

OR THREE

Example: Flip three coins, and output H if there are two heads showing, and T otherwise (this is called the "majority function," as if the coins "vote" for the outcome!)

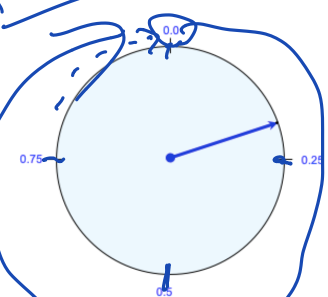
S= { H, T }

Countably Infinite

Example: Flip a coin until heads appears, and report the number of flips

S = { 1, 2, 3, 4, ... }

RANDOM()



Uncountably Infinite

Example: Spin a pointer on a circle labelled with real numbers [0..1) and report the real number that the pointer stops on.

S = [0 .. 1)

[0..1]

[0..2]

Disgression: Infinite Sets, Countable and Uncountable

When a set is Countable, it means... (wait for it).... you can *count it.*

Example 1: { A, E, I, O, U }
1, 2, 3, 4, 5

All finite sets are countable!

But you can also "count" an infinite set, if you can put it into one-to-one correspondence with the natural numbers:

BIJECTION

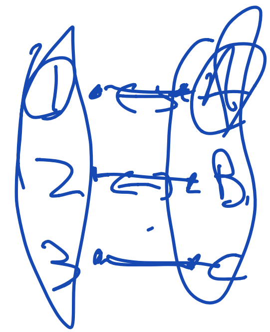
1, 2, 3, ...

∞

TH TTH TTTH

Example 2: { H, HT, HTT, HTTT, ... HT^k, ... }
1, 2, 3, 4, ... k+1, ...

$\mathbb{N}$



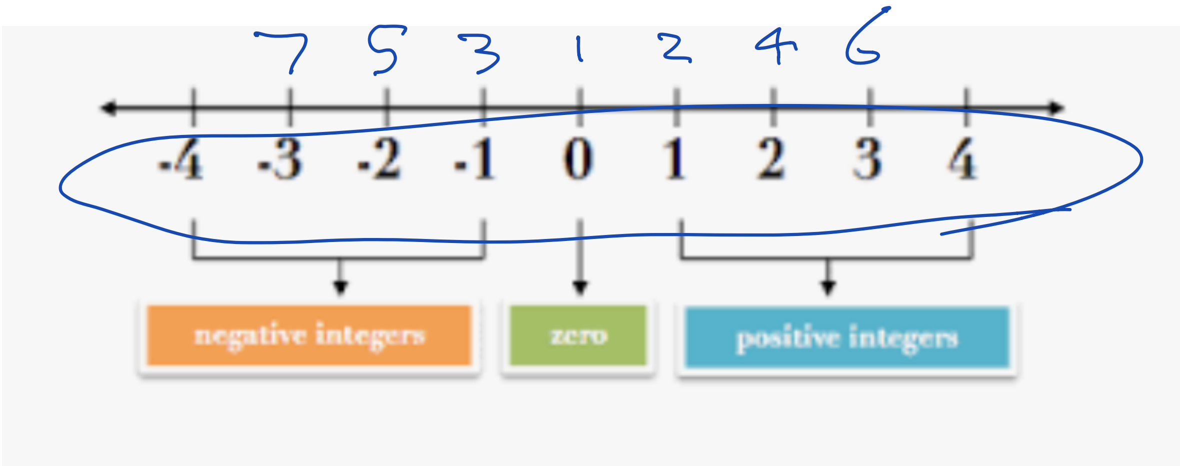
Example 3: { 0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, ... }
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, ...

Conclusion: The set of all computer programs is countable!

SET OF ALL FINITE-LENGTH BIT STRING

Disgression: Infinite Sets, Countable and Uncountable

Question: Is the set of all Integers (positive and negative and 0) countable?



Disgression: Infinite Sets, Countable and Uncountable

Question: Suppose sets S and T are both countable. Is the set of all pairs (the "cross product") S x T countable?

Example: N x N = The set of all pairs of natural numbers.

(3,4)

(287, 823)

Answer: Yes! Let S = { s_1, s_2, s_3, s_4, ... } and T = { t_1, t_2, t_3, ... }.
Then we can count the set S x Y = { (s_i, t_j) | $s_i \in$ S and $t_j \in$ T } as follows:

	s1	s2	s3	s4	....
t1	1 (s1,t1)	3 (s2,t1)	6 (s3,t1)	10 (s4,t1)	....
t2	2 (s1,t2)	5 (s2,t2)	9 (s3,t2)	(s4,t2)	....
t3	4 (s1,t3)	8 (s2,t3)	(s3,t3)	(s4,t3)	....
t4	7 (s1,t4)	(s2,t4)	(s3,t4)	(s4,t4)	....
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.

(2,3,7)

(2,(3,7))

Disgression: Infinite Sets, Countable and Uncountable

Important consequence: The set of all Rational Numbers (fractions) is countable!

$\frac{8}{0}$

	1	2	3	4	5	6	7	8	...
1	$\frac{1}{1}$ (1,1)	$\frac{1}{2}$ (1,2)	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	...
2	$\frac{2}{1}$ (2,1)	$\cancel{\frac{2}{2}}$	$\frac{2}{3}$	$\cancel{\frac{2}{4}}$	$\frac{2}{5}$	$\cancel{\frac{2}{6}}$	$\frac{2}{7}$	$\frac{2}{8}$	...
3	$\frac{3}{1}$	$\frac{3}{2}$	$\cancel{\frac{3}{3}}$	$\frac{3}{4}$	$\frac{3}{5}$	$\frac{3}{6}$	$\frac{3}{7}$	$\frac{3}{8}$	...
4	$\frac{4}{1}$	$\cancel{\frac{4}{2}}$	$\frac{4}{3}$	$\cancel{\frac{4}{4}}$	$\frac{4}{5}$	$\frac{4}{6}$	$\frac{4}{7}$	$\frac{4}{8}$	...
5	$\frac{5}{1}$	$\frac{5}{2}$	$\frac{5}{3}$	$\frac{5}{4}$	$\frac{5}{5}$	$\frac{5}{6}$	$\frac{5}{7}$	$\frac{5}{8}$	...
6	$\frac{6}{1}$	$\cancel{\frac{6}{2}}$	$\cancel{\frac{6}{3}}$	$\frac{6}{4}$	$\frac{6}{5}$	$\frac{6}{6}$	$\frac{6}{7}$	$\frac{6}{8}$	...
7	$\frac{7}{1}$	$\frac{7}{2}$	$\frac{7}{3}$	$\frac{7}{4}$	$\frac{7}{5}$	$\frac{7}{6}$	$\frac{7}{7}$	$\frac{7}{8}$	...
8	$\frac{8}{1}$	$\frac{8}{2}$	$\frac{8}{3}$	$\frac{8}{4}$	$\frac{8}{5}$	$\frac{8}{6}$	$\frac{8}{7}$	$\frac{8}{8}$	...
⋮	⋮								

$\frac{28}{24}$

$-\frac{1}{2}$

$\frac{-1}{-2}$

$\sqrt{2}$ π

Disgression: Infinite Sets, Countable and Uncountable

The last possibility, and the most subtle and confusing, is:

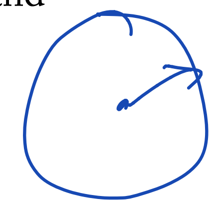
DIAGONALIZATION

G. CANTOR

Uncountably Infinite

Example: Spin a pointer on a circle labelled with real numbers [0..1) and report the number that the pointer stops on.

$S = [0 .. 1)$ $\frac{\pi}{4}$



Why is the set S uncountable?   Well, suppose it were countable, i.e, you can make a list of the infinite decimal representation of each number in S:

$(d+1)\%10$

k

1: 0. 0 0 0 0 0 0 0 0 0 0 ...
2: 0. 1 1 1 1 1 1 1 1 1 1 ...
3: 0. 0 1 0 2 0 2 0 4 0 0 ... $\to \infty$
k = 4: 0. 0 1 2 3 4 5 6 7 8 9 ...
5: 0. 1 2 3 2 1 3 5 4 2 3 ...
6: 0. 6 3 4 5 3 4 5 6 4 3 ...
7: ....
.
.
.

∞

Nope! You missed one!

0.121425...

Random Experiments, Sample Spaces and Sample Points

The major distinction we will make is between Discrete and Continuous spaces:

Discrete

Finite

Example: Flip three coins, and output head if there are two heads showing, and tails otherwise (as if the coins "vote" for the outcome!)

S= { head, tail }

Countably Infinite

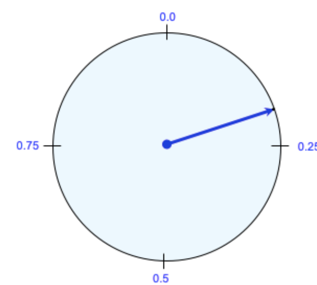
Example: Flip a coin until heads appears, and report the number of flips

S = { 1, 2, 3, 4, ... }

Continuous

Uncountably Infinite

Example: Spin a pointer on a circle labelled with real numbers [0..1) and report the number that the pointer stops on.



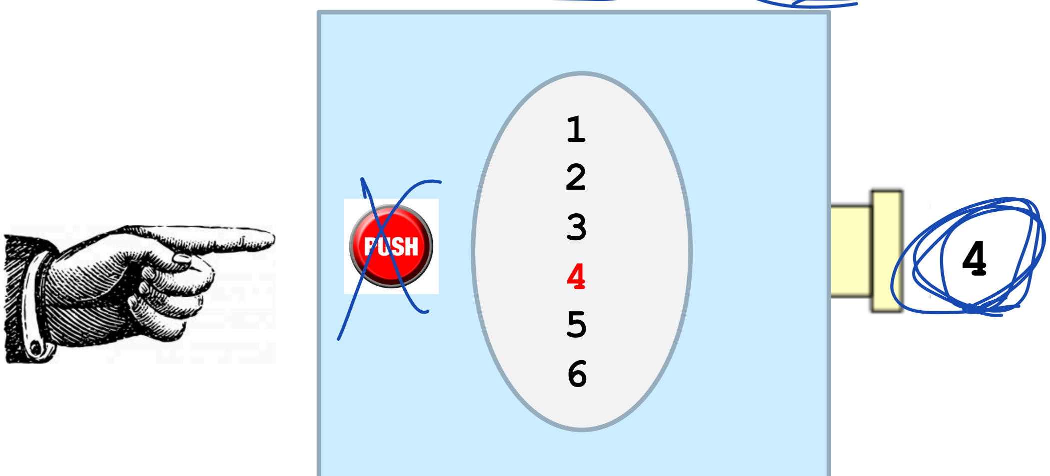
S = [0 .. 1)

Sample Spaces, Sample Points, and Events

An Event is any **subset** of the Sample Space. An event A is said to have occurred if the outcome of the random experiment is a member of A. We will be mostly interested specifying a set by some characteristic, and then calculating the probability of that event occurring.

Example: Toss a die and output the number of dots showing.
Let A = "there are an even number of dots showing" and
B = "there are at least 5 dots showing."

$S = \{ 1, 2, 3, 4, 5, 6 \}$ $A = \{ 2, 4, 6 \}$ $B = \{ 5, 6 \}$



The event A occurred, since

$4 \in \{2, 4, 6\}$

The event B did not occur:

$4 \notin \{ 5, 6 \}$

Sample Spaces, Sample Points, and Events

We will be mostly interested in **questions involving the probability of particular events occurring**, so let us pay particular attention to the notion of an event.

Example: Toss a die and output the number of dots showing. Let A = "there are an even number of dots showing."

S = { 1, 2, 3, 4, 5, 6 } A = { 2, 4, 6 } $A \subseteq S$

The set of possible events is the power-set of S, the set of all subsets,

$$\mathcal{P}(S) = \{ A \mid A \subseteq S \}$$

$\Phi \subseteq S$

$2^{|S|}$

$S \subseteq S$

So for this example we have 2^6 = 64 possible events, including

- The empty or "impossible event" $\varnothing$. ("What is the probability of rolling a 9?")
- The "certain event" S. ("What is the probability of less than 10 dots?")
- All "elementary events" of one outcome: { 1 }, { 2 } , { 3 }, ..., { 6 }.

1 2 3

etc..... This gives you the most flexible way of discussing the results of an experiment....

Sample Spaces, Sample Points, and Events

Your Turn!

Flip 3 coins and count the number of Heads showing. What is the probability of all three coins showing Tails?

You don't have to calculate the probability, but give the following:

Sample Space: $\{0, 1, 2, 3\}$

ϕ

$\{\}$

The event we are interested in (call it A):

$A = \{\cancel{0}\}$

Sample Spaces, Sample Points, and Events

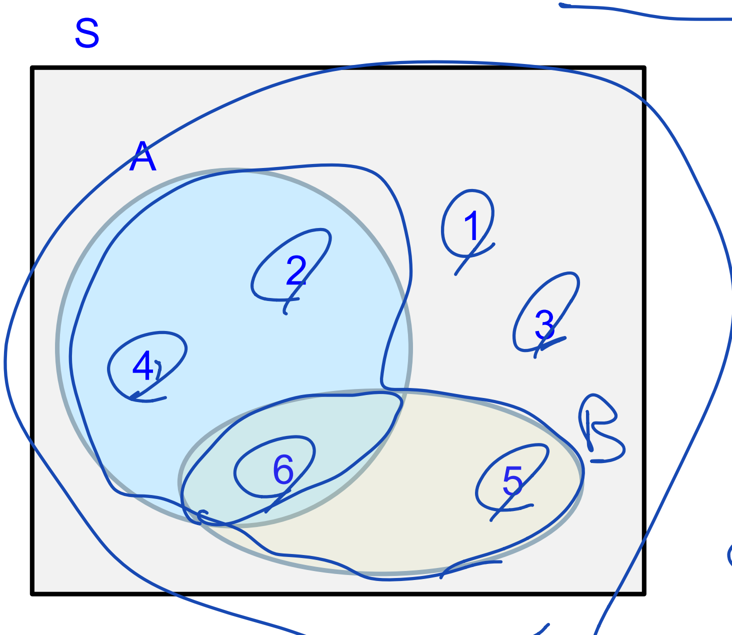
Since sample spaces are sets and events are subsets, **we can now use all the machinery of set theory to define and manipulate events.** It will also be useful to visualize events and sample spaces using Venn Diagrams:

Example: Toss a die and output the number of dots showing.
Let A = "there are an even number of dots showing" and
B = "there are at least 5 dots showing."

$A = \{2, 4, 6\}$

S = { 1, 2, 3, 4, 5, 6 }

$B = \{5, 6\}$



For any events A and B, we can define new events using set operations:

$A \cup B = \{ 2, 4, 5, 6 \}$

$A \cap B = \{ 6 \}$

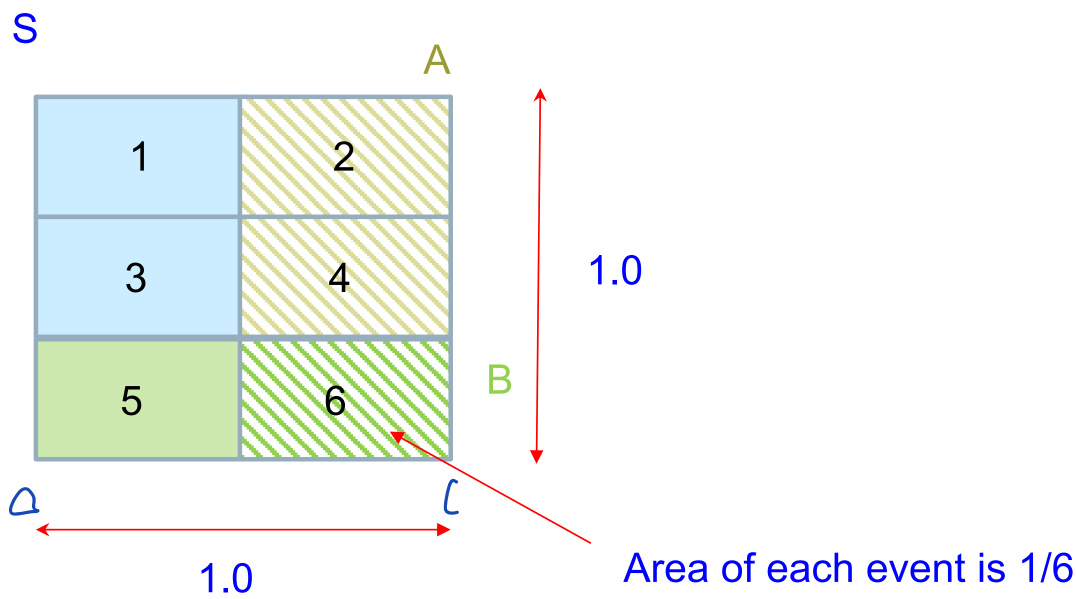
$A^c = S - A = \{ 1, 3, 5 \}$ $\bar{A}$

$A^c \cap B = \{ 5 \}$

Probability Spaces and Probability Axioms

This make perfect sense if we consider Venn Diagrams where we use area as a proxy for probability, so the area of **S = 1.0**, and the area of an event in the diagram = probability of that event.

Example: Toss a die and output the number of dots showing. Let A = "there are an even number of dots showing" and B = "there are at least 5 dots showing."



Probability Spaces and Probability Axioms

COUNT # HEADS.

FLIP 2 COINS.

WHAT IS PROB. OF AT LEAST 1 HEAD?

HH → 2

HT → 1

TH → 1

TT → $\emptyset$

$S = \{\emptyset, 1, 2\}$

$A = \{1, 2\}$

$\in A$

$B = \{2\}$

$P(B) = \frac{1}{4}$

$P(B)$

$\parallel$

$P(2) = \frac{1}{4}$

$P(A) = \frac{3}{4}$

~~$\emptyset$~~ $\{\}$

$\{0\}$

$\{1\}$.

$\{2\}$.

$\{0, 1\}$.

$\{0, 2\}$.

$\{1, 2\}$.

~~$\{0, 1, 2\}$~~.

Probability Spaces and Probability Axioms

These axioms make perfect sense if we consider Venn Diagrams where we use area as indicating probability, so the area of an event in the diagram = probability of that event.

P_1: For any event A, we have $\mathbf{P(A) \geq 0}$.

"The area of each event is non-negative."

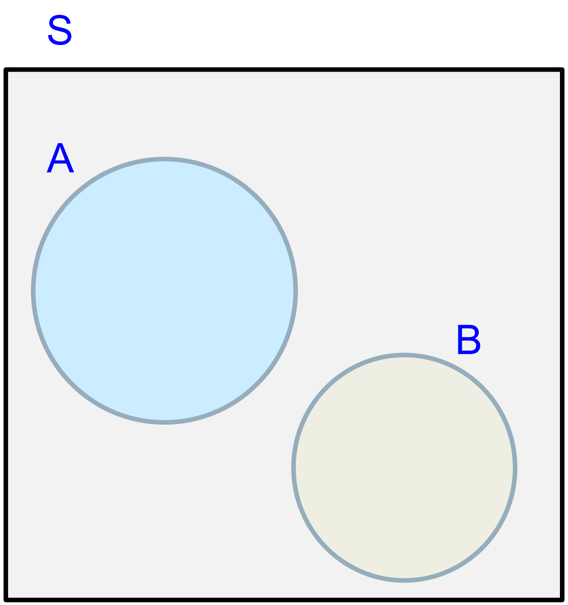
P_2: For the certain event S, we have $\mathbf{P(S) = 1.0}$.

"The area of the whole sample space is 1.0."

P_3: For any two disjoint events A and B we have

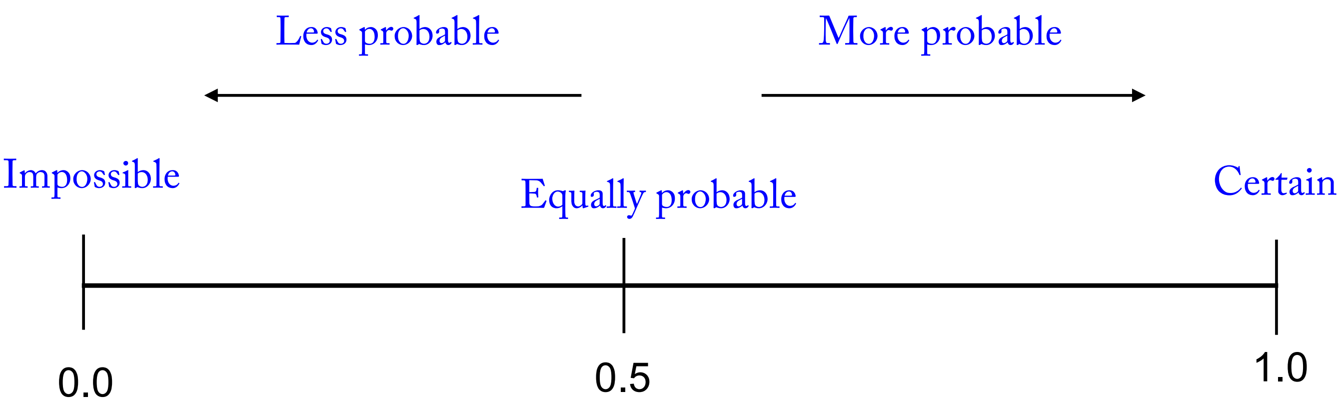
$$P(A \cup B) = P(A) + P(B)$$

"If two regions of S do not overlap, then the area of the two regions combined is the sum of the area of each region."



Probability Spaces and Probability Axioms

So we measure the probability of events on a real-number scale from 0 to 1:



toss a DIE

$S = \{1, 2, 3, 4, 5, 6\}$

$B = \{5, 6\}$